5 Practices for Orchestrating Productive Mathematics Discussions

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Includes Professional Development Guide
Introduction

As we move into the second decade of the twenty-first century, one thing is clear: Our country needs highly trained workers who can wrestle with complex problems. Gone are the days when basic skills could be counted on to yield high-paying jobs and an acceptable standard of living. Especially needed are individuals who can think, reason, and engage effectively in quantitative problem solving.

The instructional practices used in the majority of our nation’s classrooms will not prepare students for these new demands. National studies have shown that American students are not routinely asked to engage in conceptual thinking or complex problem solving (Stigler and Hiebert 1999). Most schoolwork consists of assignments composed of “problems,” for which students have been taught a preferred method of solving. There is little engagement of student “thinking” in such tasks, only the straightforward application of previously learned skills and recall of memorized facts. It is unrealistic to expect students to learn to grapple with the unstructured, messy challenges of today’s world if they are forced to sit silently in rows, complete basic skills worksheets, and engage in teacher-led “discussions” that consist of literal, fact-based questions and answers.

What kind of learning experiences will prepare students for the demands of the twenty-first century? Research tells us that complex knowledge and skills are learned through social interaction (Vygotsky 1978; Lave and Wenger 1991). We learn through a process of knowledge construction that requires us to actively manipulate and refine information and then integrate it with our prior understandings. Social interaction provides us with the opportunity to use others as resources, to share our ideas with others, and to participate in the joint construction of knowledge. In mathematics classrooms, high-quality discussions support student learning of mathematics by helping students learn how to communicate their ideas, making students’ thinking public so it can be guided in mathematically sound directions, and encouraging students to evaluate their own and each other’s mathematical ideas. These are all important features of what it means to be “mathematically literate.”

Creating discussion-based opportunities for student learning will require learning on the part of many teachers. First, teachers will need to learn how to select and set up cognitively challenging instructional tasks in their classrooms, since such high-level tasks provide the grist for worthwhile discussions. Over the years, however, most textbooks have fed teachers a steady diet of routine, procedural tasks around which it would be difficult, if not impossible, to organize an engaging discussion.

Second, teachers must learn how to support their students as they engage with and discuss their solutions to cognitively challenging tasks. We know from our own past research that once high-level tasks are introduced in the classroom, many teachers have difficulty maintaining the cognitive demand of those tasks as students engage with them (Stein, Grover, and Henningsen 1996). Students often end up thinking and reasoning at a lower level than the task is intended to elicit. One of the reasons for this is teachers’ difficulties in orchestrating discussions that productively use students’ ideas and strategies that are generated in response to high-level tasks.

A typical lesson that uses a high-level instructional task proceeds in three phases. It begins with the teacher’s launching of a mathematical problem that embodies important mathematical ideas and can be solved in multiple ways. During this “launch phase,” the teacher introduces students to the problem, the tools that are available for working on it, and the nature of the products that the students will be expected to produce. This phase is followed by the “explore phase,” in which students work on the problem, often discussing it in pairs or small groups. As students work on the problem, they are encouraged to solve it in
whatever way makes sense to them and be prepared to explain their approach to others in the class. The lesson then concludes with a whole-class discussion and summary of various student-generated approaches to solving the problem. During this “discuss and summarize” phase, a variety of approaches to the problem are displayed for the whole class to view and discuss.

Why are these end-of-class discussions so difficult to orchestrate? Research tells us that students learn when they are encouraged to become the authors of their own ideas and when they are held accountable for reasoning about and understanding key ideas (Engle and Conant 2002). In practice, doing both of these simultaneously is very difficult. By their nature, high-level tasks do not lead all students to solve the problem in the same way. Rather, teachers can and should expect to see varied (both correct and incorrect) approaches to solving the task during the discussion phase of the lesson. In theory, this is a good thing because students are “authoring” (or constructing) their own ways of solving the problem.

The challenge rests in the fact that teachers must also align the many disparate approaches that students generate in response to high-level tasks with the learning goal of the lesson. It is the teachers’ responsibility to move students collectively toward, and hold them accountable for, the development of a set of ideas and processes that are central to the discipline—those that are widely accepted as worthwhile and important in mathematics as well as necessary for students’ future learning of mathematics in school. If the teacher fails to do this, the balance tips too far toward student authority, and classroom discussions become unmoored from accepted disciplinary understandings.

The key is to maintain the right balance. Too much focus on accountability can undermine students’ authority and sense making and, unwittingly, encourage increased reliance on teacher direction. Students quickly get the message—often from subtle cues—that “knowing mathematics” means using only those strategies that have been validated by the teacher or textbook; correspondingly, they learn not to use or trust their own reasoning. Too much focus on student authorship, on the other hand, leads to classroom discussions that are free-for-alls.

**Successful or Superficial? Discussion in David Crane’s Classroom**

In short, the teacher’s role in discussions is critical. Without expert guidance, discussions in mathematics classrooms can easily devolve into the teacher taking over the lesson and providing a “lecture,” on the one hand, or, on the other, the students presenting an unconnected series of show-and-tell demonstrations, all of which are treated equally and together illuminate little about the mathematical ideas that are the goal of the lesson. Consider, for example, the following vignette (from Stein and colleagues [2008]), featuring a fourth-grade teacher, David Crane.

**ACTIVE ENGAGEMENT 0.1**
As you read the Case of David Crane, identify instances of student authorship of ideas and approaches, as well as instances of holding students accountable to the discipline.
Leaves and Caterpillars: The Case of David Crane

Students in Mr. Crane's fourth-grade class were solving the following problem: "A fourth-grade class needs 5 leaves each day to feed its 2 caterpillars. How many leaves would the students need each day for 12 caterpillars?" Mr. Crane told his students that they could solve the problem any way they wanted, but he emphasized that they needed to be able to explain how they got their answer and why it worked.

As students worked in pairs to solve the problem, Mr. Crane walked around the room, making sure that students were on task and making progress on the problem. He was pleased to see that students were using many different approaches to the problem—making tables, drawing pictures, and, in some cases, writing explanations.

He noticed that two pairs of students had gotten wrong answers (see fig. 0.1). Mr. Crane wasn't too concerned about the incorrect responses, however, since he felt that once several correct solution strategies were presented, these students would see what they did wrong and have new strategies for solving similar problems in the future.

![Fig. 0.1. Solutions produced by Darnell and Marcus (left) and Missy and Kate (right)](image)

When most students were finished, Mr. Crane called the class together to discuss the problem. He began the discussion by asking for volunteers to share their solutions and strategies, being careful to avoid calling on the students with incorrect solutions. Over the course of the next 15 minutes, first Kyra, then Jason, Jamal, Melissa, Martin, and Janine volunteered to present the solutions to the task that they and their partners had created (see fig. 0.2). During each presentation, Mr. Crane made sure to ask each presenter questions that helped the student to clarify and justify the work. He concluded the class by telling students that the problem could be solved in many different ways and now, when they solved a problem like this, they could pick the way they liked best because all the ways gave the same answer.
<table>
<thead>
<tr>
<th>Janine's Work</th>
<th>Kyra's Work</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Answer:</strong> 30</td>
<td><strong>Answer:</strong> 30</td>
</tr>
<tr>
<td>if each of the caterpillars needs 2 1/2 leaves a day then you just x 2 1/2 x 12 = 30.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Jamal's Work</th>
<th>Martin's Work</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Answer:</strong> 30 leaves</td>
<td><strong>Answer:</strong> 30 leaves</td>
</tr>
<tr>
<td>5</td>
<td>2 1/2</td>
</tr>
<tr>
<td>2</td>
<td>2 1/2</td>
</tr>
<tr>
<td>2</td>
<td>2 1/2</td>
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<tr>
<td>2</td>
<td>2 1/2</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Jason's Work</th>
<th>Melissa's Work</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Answer:</strong> 30</td>
<td><strong>Answer:</strong> 30</td>
</tr>
<tr>
<td>If it takes 5 leaves for two caterpillars, you just count by 5s, until you come to half of 12. The number is 30, and then you multiply 5x6 and it equals 30.</td>
<td></td>
</tr>
</tbody>
</table>

*Fig. 0.2. Solutions shared by students in Mr. Crane's class*
Analyzing the Case of David Crane

Some would consider Mr. Crane's lesson exemplary. Indeed, Mr. Crane did many things well, including allowing students to construct their own way of solving this cognitively challenging task and stressing the importance of students' being able to explain their reasoning. Students were working with partners and publicly sharing their solutions and strategies with their peers; their ideas appeared to be respected. All in all, students in Mr. Crane's class had the opportunity to become the "authors" of their own knowledge of mathematics.

However, a more critical eye might have noted that the string of presentations did not build toward important mathematical ideas. The upshot of the discussion appeared to be "the more ways of solving the problem, the better," but, in fact, Mr. Crane held each student accountable for knowing only one way to solve the problem. In addition, although Mr. Crane observed students as they worked, he did not appear to use this time to assess what students understood about proportional reasoning or to select particular students' work to feature in the whole-class discussion. Furthermore, he gathered no information regarding whether the two pairs of students who had gotten the wrong answer (Darnell and Marcus, and Missy and Kate) were helped by the student presentations of correct strategies. Had they diagnosed the faulty reasoning in their approaches?

In fact, we argue that much of the discussion in Mr. Crane's classroom was show-and-tell, in which students with correct answers each take turns sharing their solution strategies. The teacher did little filtering of the mathematical ideas that each strategy helped to illustrate, nor did he make any attempt to highlight those ideas. In addition, the teacher did not draw connections among different solution methods or tie them to important disciplinary methods or mathematical ideas. Finally, he gave no attention to weighing which strategies might be most useful, efficient, accurate, and so on, in particular circumstances. All were treated as equally good.

In short, providing students with cognitively demanding tasks with which to engage and then conducting show-and-tell discussions cannot be counted on to move an entire class forward mathematically. Indeed, this kind of practice has been criticized for creating classroom environments in which nearly complete control of the mathematical agenda is relinquished to students. Some teachers misperceive the appeal to honor students' thinking and reasoning as a call for a complete moratorium on teachers' shaping of the quality of students' mathematical thinking. As a result of the lack of guidance with respect to what teachers could do to encourage rigorous mathematical thinking and reasoning, many teachers were left feeling that they should avoid telling students anything.

A related criticism of inquiry-oriented lessons concerns the fragmented and often incoherent nature of the discuss-and-summarize phases of lessons. In these show-and-tells, as exemplified in David Crane's classroom, one student presentation would follow another with limited teacher (or student) commentary and no assistance with respect to drawing connections among the methods or tying them to widely shared disciplinary methods and concepts. The discussion offered no mathematical or other reason for students to necessarily listen to or try to understand the methods of their classmates. As illustrated in Mr. Crane's comment at the end of the class, students could simply "pick the way they liked best." This type of situation has led to an increasingly recognized dilemma associated with inquiry- and discovery-based approaches to teaching: the challenge of aligning students' developing ideas and methods with the disciplinary ideas that they ultimately are accountable for knowing.
In sum, David Crane did little to encourage accountability to the discipline of mathematics. How could he have more firmly supported student accountability without undermining student authority? The single most important thing that he could have done would be to have set a clear goal for what he wanted students to learn from the lesson. Without a learning objective in mind, the various solutions that were presented, although all correct, were scattered in the “mathematical landscape.” If, however, he had targeted the learning goal of, for example, making sure that all students recognized that the relationship between caterpillars and leaves was multiplicative and not additive, he might have monitored students’ work with this in mind. Whose work illustrated the multiplicative relationship particularly well? Did the students’ work include examples of different ways of illustrating this relationship—examples that could connect with known mathematical strategies (e.g., unit rate, scaling up)? This assessment of student work would have allowed him to be more deliberate about which students he selected to present during the discussion phase. He might even have wanted to have the incorrect, additive solutions displayed so that students could recognize the faulty reasoning that underlie them. With an array of purposefully selected strategies presented, Mr. Crane would then be in a position to steer the discussion toward a more mathematically satisfying conclusion.

Conclusion

The Case of David Crane illustrates the need for guidance in shaping classroom discussions and maximizing their potential to extend students’ thinking and connect it to important mathematical ideas. The chapters that follow offer this guidance by elaborating a practical framework, based on five doable instructional practices, for orchestrating and managing productive classroom discussions.
Introducing the Five Practices

Many teachers are daunted by an approach to pedagogy that builds on student thinking. Some are worried about content coverage, asking, “How can I be assured that students will learn what I am responsible for teaching if I don’t march through the material and tell them everything they need to know?” Others—teachers who perhaps are already convinced of the importance of student thinking—may be nonetheless worried about their ability to diagnose students’ thinking on the fly and to quickly devise responses that will guide students to the correct mathematical understanding.

Teachers are correct when they acknowledge that this type of teaching is demanding. It requires knowledge of the relevant mathematical content, of student thinking about that content, and of the subtle pedagogical “moves” that a teacher can make to lead discussions in fruitful directions, along with the ability to rapidly apply all of this in specific circumstances. Yet, we have seen many teachers learn to teach in this way, with the help of the five practices.

We think of the five practices as skillful improvisation. The practices that we have identified are meant to make student-centered instruction more manageable by moderating the degree of improvisation required by the teacher during a discussion. Instead of focusing on in-the-moment responses to student contributions, the practices emphasize the importance of planning. Through planning, teachers can anticipate likely student contributions, prepare responses that they might make to them, and make decisions about how to structure students’ presentations to further their mathematical agenda for the lesson. We turn now to an explication of the five practices.

The Five Practices

The five practices were designed to help teachers to use students’ responses to advance the mathematical understanding of the class as a whole by providing teachers with some control over what is likely to happen in the discussion as well as more time to make instructional decisions by shifting some of the decision making to the planning phase of the lesson. The five practices are—
1. **anticipating** likely student responses to challenging mathematical tasks;
2. **monitoring** students' actual responses to the tasks (while students work on the tasks in pairs or small groups);
3. **selecting** particular students to present their mathematical work during the whole-class discussion;
4. **sequencing** the student responses that will be displayed in a specific order; and
5. **connecting** different students' responses and connecting the responses to key mathematical ideas.

Each practice is described in more detail in the following sections, which illustrate them by identifying what Mr. Crane *could have done* in the Leaves and Caterpillars lesson (presented in the introduction), to move student thinking more skillfully toward the goal of recognizing that the relationship between caterpillars and leaves is multiplicative, not additive.

**Anticipating**

The first practice is to make an effort to actively envision how students might mathematically approach the instructional task or tasks that they will work on. This involves much more than simply evaluating whether a task is at the right level of difficulty or of sufficient interest to students, and it goes beyond considering whether or not they are getting the "right" answer.

Anticipating students' responses involves developing considered expectations about how students might mathematically interpret a problem, the array of strategies—both correct and incorrect—that they might use to tackle it, and how those strategies and interpretations might relate to the mathematical concepts, representations, procedures, and practices that the teacher would like his or her students to learn.

Anticipating requires that teachers do the problem as many ways as they can. Sometimes teachers find that it is helpful to expand on what they might be able to think of individually by working on the task with colleagues, reviewing responses to the task that might be available (e.g., work produced by students in the previous year, responses that are published along with tasks in supplementary materials), and consulting research on student learning of the mathematical ideas embedded in the task. For example, research suggests that students often use additive strategies (such as Missy and Kate's response, shown in fig. 0.1) to solve tasks like the Leaves and Caterpillars problem, in which there is a multiplicative relationship between quantities (Hart 1981; Heller et al. 1989; Kaput and West 1994). Anticipating this approach in advance of the lesson would have made it possible for Mr. Crane to recognize it when his students produced it and carefully consider what actions he might take should they do so (e.g., what questions to ask so that students become aware of the multiplicative nature of the relationship between the caterpillars and leaves, how to bring up the solution during discussion so that all students might consider why it is not a valid method).

In addition, if Mr. Crane had solved the problem ahead of time in as many ways as possible, he might have realized that there were at least two different strategies for arriving at the correct answer—unit rate and scale factor—and that each of these could be expressed with different representations (pictures, tables, and written explanations).
Monitoring

Monitoring student responses involves paying close attention to students’ mathematical thinking and solution strategies as they work on the task. Teachers generally do this by circulating around the classroom while students work either individually or in small groups. Carefully attending to what students do as they work makes it possible for teachers to use their observations to decide what and whom to focus on during the discussion that follows (Lampert 2001).

One way to facilitate the monitoring process is for the teacher, before beginning the lesson, to create a list of solutions that he or she anticipates that students will produce and that will help in accomplishing his or her mathematical goals for the lesson. The list, such as the one shown in column 1 of the chart in figure 1.1 for the Leaves and Caterpillars task, can help the teacher keep track of which students or groups produced which solutions or brought out which ideas that he or she wants to make sure to capture during the whole-group discussion. The “Other” cell in the first column provides the teacher with the opportunity to capture ideas that he or she had not anticipated.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Who and What</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unit rate</td>
<td>Janine – multiplied 12 × 2.5 (sticks representing caterpillars)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Kyra – added 2.5 12 times (picture of leaves and caterpillars)</td>
<td></td>
</tr>
<tr>
<td>Scale Factor</td>
<td>Jason – narrative description</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(12) is 6 times the original amount (2), so the number of leaves (30) must be</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6 times the original amount (5)</td>
<td></td>
</tr>
<tr>
<td>Scaling Up</td>
<td>Jamal – table with leaves and caterpillars increasing in increments of 2 and</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Missy and Kate</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(10) (2 + 10 = 12), so the number of leaves must also increase by 10 (5 + 10 = 15)</td>
<td></td>
</tr>
<tr>
<td>Additive</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Other</td>
<td>Martin (picture)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Melissa (table)</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 1.1. A chart for monitoring students’ work on the Leaves and Caterpillars task

As discussed in the introduction, Mr. Crane’s lesson provided limited, if any, evidence of active monitoring. Although Mr. Crane knew who got correct answers and who did not and that a range of strategies had been used, his choice of students to present at the end of the class suggests that he had not monitored the specific mathematical learning potential available in any of the responses.

What Mr. Crane could have done while students worked on the task is shown in the second column in the chart in figure 1.1.
It is important to note, however, that monitoring involves more than just watching and listening to students. During this time, the teacher should also ask questions that will make students’ thinking visible, help students clarify their thinking, ensure that members of the group are all engaged in the activity, and press students to consider aspects of the task to which they need to attend. Many of these questions can be planned in advance of the lesson, on the basis of the anticipated solutions. For example, if Mr. Crane had anticipated that a student would use a unit-rate approach (Janine’s or Kyra’s responses—see fig. 1.2), reasoning from the fact that the number of leaves eaten by one caterpillar was 2.5, then he might have been prepared to question, say, for example, Janine, regarding how she came up with the number 2.5 and how she knew to multiply it by 12. Questioning a student or group of students while they are exploring the task provides them with the opportunity to refine or revise their strategy prior to whole-group discussion and provides the teacher with insights regarding what the student understands about the problem and the mathematical ideas embedded in it.

**Selecting**

Having monitored the available student strategies in the class, the teacher can then select particular students to share their work with the rest of the class to get specific mathematics into the open for examination, thus giving the teacher more control over the discussion (Lampert 2001). The selection of particular students and their solutions is guided by the mathematical goal for the lesson and the teacher’s assessment of how each contribution will contribute to that goal. Thus, the teacher selects certain students to present because of the mathematics in their responses.

A typical way to accomplish “selection” is to call on specific students (or groups of students) to present their work as the discussion proceeds. Alternatively, the teacher may let students know before the discussion that they will be presenting their work. In a hybrid variety, a teacher might ask for volunteers but then select a particular student that he or she knows is one of several who have a particularly useful idea to share with the class. By calling for volunteers but then strategically selecting from among them, the teacher signals appreciation for students’ spontaneous contributions, while at the same time keeping control of the ideas that are publicly presented.

Returning to the Leaves and Caterpillar vignette, if we look at the strategies that were shared, we note that Kyra and Janine had similar strategies that used the idea of unit rate (i.e., finding out the number of leaves needed for one caterpillar). Given that, there may not have been any added mathematical value to sharing both. In fact, if Mr. Crane wanted to students to see the multiplicative nature of the relationship, he might have selected Janine, since her approach clearly involved multiplication.

Also, there might have been some payoff from sharing the solution produced by Missy and Kate (fig. 0.1) and contrasting it with the solution produced by Melissa (fig. 0.2). Although both approaches used addition, Missy and Kate inappropriately added the same number (10) to both the leaves and the caterpillars. Melissa, on the other hand, added 5 leaves for every 2 caterpillars, illustrating that she understood that this ratio (5 for every 2) had to be kept constant.

**Sequencing**

Having selected particular students to present, the teacher can then make decisions regarding how to sequence the student presentations. By making purposeful choices about the order in which
students’ work is shared, teachers can maximize the chances of achieving their mathematical goals for the discussion. For example, the teacher might want to have the strategy used by the majority of students presented before those that only a few students used, to validate the work that the majority of students did and make the beginning of the discussion accessible to as many students as possible. Alternatively, the teacher might want to begin with a strategy that is more concrete (using drawings or concrete materials) and move to strategies that are more abstract (using algebra). This approach—moving from concrete to abstract—serves to validate less sophisticated approaches and allows for connections among approaches. If a common misconception underlies a strategy that several students used, the teacher might want to have it addressed first so that the class can clear up that misunderstanding to be able to work on developing more successful ways of tackling the problem. Finally, the teacher might want to have related or contrasting strategies presented one right after the other in order to make it easier for the class to compare them. Again, during planning the teacher can consider possible ways of sequencing anticipated responses to highlight the mathematical ideas that are key to the lesson. Unanticipated responses can then be fitted into the sequence as the teacher makes final decisions about what is going to be presented.

More research needs to be done to compare the value of different sequencing methods, but we want to emphasize here that particular sequences can be used to advance particular goals for a lesson. Returning to the Leaves and Caterpillar vignette, we point out one sequence that could have been used: Martin (scaling up by collecting sets—picture), Jamal (scaling up—table), Janine (unit rate—picture/written explanation); and Jason (scale factor—written explanation).

This ordering begins with the least sophisticated representation (a picture) of the least sophisticated strategy (scaling up by collecting sets) and ends with the most sophisticated strategy (scale factor), a sequencing that would help with the goal of accessibility. In addition, by having the same strategy (scaling up) embodied in two different representations (a picture and a table), students would have the opportunity to develop deeper understandings of how to think about this problem in terms of scaling up.

**Connecting**

Finally, the teacher helps students draw connections between their solutions and other students’ solutions as well as the key mathematical ideas in the lesson. The teacher can help students to make judgments about the consequences of different approaches for the range of problems that can be solved, one’s likely accuracy and efficiency in solving them, and the kinds of mathematical patterns that can be more easily discerned. Rather than having mathematical discussions consist of separate presentations of different ways to solve a particular problem, the goal is to have student presentations build on one another to develop powerful mathematical ideas.

Returning to Mr. Crane’s class, let’s suppose that the sequencing of student presentations was Martin, Jamal, Janine, and Jason, as discussed above. Students could be asked to compare Jamal and Janine’s responses and to identify where Janine’s unit rate (2.5 leaves per caterpillar) is found in Jamal’s table (it is the factor by which the number of caterpillars must be multiplied to get the number of leaves). Students could also be asked to compare Jason’s explanation with Jamal and Martin’s work to see if the scale factor of 6 can be seen in each of their tabular and pictorial representations.

It is important to note that the five practices build on another. Monitoring is less daunting if the teacher has taken the time to anticipate ways in which students might solve a task. Although
a teacher cannot know with 100 percent certainty how students will solve a problem prior to the lesson, many solutions can be anticipated and thus easily recognized during monitoring. A teacher who has already thought about the mathematics represented by those solutions can turn his or her attention to making mathematical sense of those solutions that are unanticipated. Selecting, sequencing, and connecting, in turn, build on effective monitoring. Effective monitoring will yield the substance for a discussion that builds on student thinking, yet moves assuredly toward the mathematical goal of the lesson.

Conclusion

The purpose of the five practices is to provide teachers with more control over student-centered pedagogy. They do so by allowing the teacher to manage the content that will be discussed and how it will be discussed. Through careful planning, the amount of improvisation required by the teacher “in the moment” is kept to a minimum. Thus, teachers are freed up to listen to and make sense of outlier strategies and to thoughtfully plan connections between different ways of solving problems. All of this leads to more coherent, yet student-focused, discussions.

In the next chapter, we explore an important first step in enacting the five practices: setting goals for instruction and identifying appropriate tasks. Although this work is not one of the five practices, it is the foundation on which the five practices are built. In chapters 3, 4, and 5, we then explore the five practices in depth and provide additional illustrations showing what the practices look like when enacted and how the practices can lead to more productive discussions.
Investigating the Five Practices in Action

In chapter 1, we presented the five practices for orchestrating a productive discussion and considered what David Crante's class might have looked like had he engaged in these practices and how use of the practices in advance of and during the lesson could have had an impact on students' opportunities to learn mathematics. In this chapter, we analyze the teaching of Darcy Dunn, an eighth-grade teacher who has spent several years trying to improve the quality of discussions in her classroom.

The Five Practices in the Case of Darcy Dunn

The vignette that follows, Tiling a Patio: The Case of Darcy Dunn, provides an opportunity to consider the extent to which the teacher appears to have engaged in some or all of the five practices before or during the featured lesson and the ways in which her use of the practices may have contributed to students' opportunities to learn. (This case, written by Smith, Hillen, and Catania [2007], is based on observed instruction in the third author's classroom.)

ACTIVE ENGAGEMENT 3.1
Read the vignette Tiling a Patio: The Case of Darcy Dunn and identify places in the lesson where Ms. Dunn appears to be engaging in the five practices. Use the line numbers to help you keep track of the places where you think she used each practice.

Tiling a Patio: The Case of Darcy Dunn
Darcy Dunn was working on a unit on functions with her eighth-grade students early in the school year and decided to engage them in solving the Tiling a Patio task (shown previously as fig. 2.3 but repeated here as fig. 3.1 for the reader's convenience). As a
result of this lesson, she wanted her students to understand three mathematical ideas: (1) that linear functions grow at a constant rate; (2) that there are different but equivalent ways of writing an explicit rule that defines the relationship between two variables; and (3) that the rate of change of a linear function can be highlighted in different representational forms: as the successive difference in a table of \((x, y)\) values in which values for \(x\) increase by 1, as the \(m\) value in the equation \(y = mx - b\), and as the slope of the function when graphed.

Alfredo Gomez is designing patios. Each patio has a rectangular garden area in the center. Alfredo uses black tiles to represent the soil of the garden. Around each garden, he designs a border of white tiles. The pictures shown below show the three smallest patios that he can design with black tiles for the garden and white tiles for the border.

![Patio Diagrams](image)

a. Draw patio 4 and patio 5. How many white tiles are in patio 4? Patio 5?
b. Make some observations about the patios that could help you describe larger patios.
c. Describe a method for finding the total number of white tiles needed for patio 50 (without constructing it).
d. Write a rule that could be used to determine the number of white tiles needed for any patio. Explain how your rule relates to the visual representation of the patio.
e. Write a different rule that could be used to determine the number of white tiles needed for any patio. Explain how your rule relates to the visual representation of the patio.

Fig. 3.1. The Tiling a Patio task. (Adapted from Cuevas and Yeatts [2005, pp. 18-22].)

In addition to the fact that the task provided a context for exploring the mathematical ideas that Ms. Dunn had targeted, it had an aspect that she found particularly appealing: all students, regardless of prior knowledge and experiences, would have access to the task. Every student would be able to build or draw the next two patios (part d) and make some observations about the patios (part e).

Although finding the number of white (border) tiles in patio 50 (part c) would be more challenging, students could make a table and look for numeric patterns or “see” one of the many relationships between the white and black tiles in the diagram itself.

Ms. Dunn began the lesson by having a student read the task aloud and making sure that all of the students understood what the problem was asking. She told students that they would have five minutes of “private think time” to begin working on the problem individually and reminded them to help themselves to any of the materials (tiles, grid paper, colored pencils, calculators) on their tables. They could then share their ideas about the task with the other members of their table groups and work together to come up with a solution.
As students worked on the task, first on their own and later in collaboration with their peers, Ms. Dunn circulated among the groups, making note of the different approaches that students were using, asking clarifying questions, and pressing the students to think about what the bigger patios would "look like" and how they could figure that out without building or drawing them all. She noted that although all the groups were able to complete parts a and b, a few students, such as James, were having difficulty describing patio 50, and many were struggling to write symbolic rules for part c. Through her questioning during small-group work, these struggling students had made some progress, and she decided that the students could continue working on providing verbal descriptions and converting them to symbolic rules as a whole class.

After about fifteen minutes of small-group work, Ms. Dunn decided that she would ask Beth to present her group's strategy first for part d. Several groups had used the same approach, but it had been several days since Beth had contributed to a whole-class discussion in a central way, and Ms. Dunn wanted this quiet student to have a chance to demonstrate her competence. As Beth approached the overhead projector in the front of the room, Ms. Dunn handed her a few overhead pens in different colors and one of the transparencies that she had prepared in advance, showing the first three patios. This way, Beth could easily explain what she did and how it connected to the drawing without having to draw all the patios. The following dialogue ensued between Beth and Ms. Dunn:

Beth: You multiply by two and add six.
Ms. Dunn: You multiply what by two?
Beth: The black tiles.
Ms. Dunn: Write it down somewhere. You multiply the black tiles by two, and then add six. Can you show us on the diagram—where do you see it on the picture? Where do you see that, to multiply by two? You can write on the transparency.
Beth: [Demonstrating her method on the drawing of patio 1] There's one, then one tile times two equals two, plus six, equals eight, and then, it's eight tiles.
Ms. Dunn: OK, you add six. Where is the constant of six?
Beth: Because there's three on each side.
Ms. Dunn: Circle them for me.
Beth: [Makes circles around the tiles on the sides of patio 1, as shown in fig. 3.2a.]
Ms. Dunn: One, and the two—where's the two? Two ones are where?
Beth: Right there, and right there [points to the middle tile of the three tiles on the top row and the bottom row of patio 1, as shown in fig. 3.2b.]
After Beth's presentation, Ms. Dunn pressed students to express Beth's way of viewing the pattern symbolically as \( t = 2b + 6 \), where \( t \) is the number of white tiles in the patio and \( b \) is the number of black tiles. Sherrill commented that the number of black tiles was the same as the patio number, so it didn't matter if they used \( b \) (for black tiles) or \( p \) (for patio number). Ms. Dunn asked Sherrill to write the generalization for the number of tiles on the newsprint that was hanging on the board so that everyone could keep track of the different ways of finding the total number of white tiles in any patio.

Ms. Dunn then asked for a second method from the class. Several students volunteered to present their work, and after quickly checking the notes that she had made as she had monitored the small-group work, Ms. Dunn selected Faith to go next. On a new transparency, Faith demonstrated her approach to patio 1 [shown in fig. 3.3], explaining, "I did the number of black tiles, and I added two [see step 1 in fig. 3.3]. You do that times two to get the top and bottom [see step 2 in fig. 3.3]. Then I did plus two" [see step 3 in fig. 3.3].
When Faith had finished her explanation, Ms. Dunn commented, “OK, I don’t think everyone understood that. Does anyone have a question for Faith?” Pedro was the first to raise his hand, and Ms. Dunn encouraged Faith to call on him. Pedro asked, “Where did your last ‘plus two’ come from?” Faith clarified, “These two right here [pointing to the white tiles to the left and right of the black tile in patio 1], because they’re the two remaining tiles that you haven’t added already.”

Ms. Dunn then asked the class how they could write an equation for Faith’s approach. Damien volunteered that his group had thought about the problem in the same way that Faith did, and they had come up with the equation $w = 2(b + 2) + 2$. At the teacher’s request, Damien went to the front of the room to explain why this equation worked, using the drawing of patio 3. He explained, “The number of black tiles (or the patio number) plus two will always give you the top and bottom rows, and then you always have one on each side which gives you the plus two.” Ms. Dunn asked Damien to add the equation to the newsprint list.

Ms. Dunn then asked Devon if he would be willing to share his approach. Time was running out, and Ms. Dunn wanted to make sure that his approach, which focused on finding the total area of the rectangular region (the patio plus the garden) and then subtracting out the area of the garden, was made public, since it was different from other approaches and had the potential to be a useful strategy for solving problems that students would encounter in the future. Ms. Dunn engaged Devon in the following dialogue:

Devon: OK, like Damien was saying, there’s always going to be two more tiles on the bottom [row].

Ms. Dunn: Draw on it [hands Damien a transparency].

Devon: [Drawing and explaining] There’s always going to be two more tiles down here [see step 1 in fig. 3.4] than there is right here. So, I knew that in patio 50 there was going to be fifty-two on the bottom, cause there’s fifty black tiles. And, so I took fifty-two times three, these three [pointing], ‘cause there’s always three on the side [see step 2 in fig. 3.4], no matter what patio it is, and I got a hundred fifty-six. Which gives you the area; then you subtract the black ones [see step 3 in fig. 3.4], so you subtract fifty and that gives you a hundred and six.

Ms. Dunn: Oh! That was pretty creative. He took the whole figure, and then subtracted out the area in the middle. Ooh—I like it.

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Fig. 3.4. Devon’s explanation of his approach to patio 2
Ms. Dunn then asked the class how they could write Devon's rule, using symbols. Phoebe said it would be $w = 3(b + 2) - b$. James had a puzzled look on his face, and Ms. Dunn asked him if he had a question for Phoebe. James asked, "Why did you multiply by 3? Everyone else multiplied by 2." Phoebe responded, "Devon is using all three rows of the patio, so he has three rows of $p + 2$, not two rows like Faith had. But then, you have to subtract the black part because it isn't part of the patio." James said, "So you took times three and subtracted, while Faith did times two and added. I get it."

Ms. Dunn very much wanted students to consider the table that Tamika had built when she started the problem. She thought that this representation, which included the first ten patios, would help students see that the number of white tiles increased by 2 as the patio number increased by 1 (that is, that the rate of change is 2)—an idea that had not been salient in any of the presentations so far. Ms. Dunn then planned to ask students to show where this "+2" was in the picture and in the equation. She wanted to make sure that they saw the connection among the picture, the table, and the equation. She then wanted to have students predict what the graph would look like and why, and, ultimately, graph it. But she knew that this work could not be done in the remaining five minutes of class.

Instead, she decided that she would begin tomorrow's class with a discussion of the table and the graph.

Ms. Dunn decided to use the limited time she had left to return to the list of equations that the students had produced during the discussion, to which Phoebe had added the last equation. She called the students' attention to the list that was hanging in the front of the room (shown in fig. 3.5) and noted, "We came up with three different ways to find the total number of white tiles in any patio. Can they all be right?" She then asked students to spend the next few minutes discussing this question in their groups. Their homework assignment was to provide a written answer to the question and to justify their conclusion.

In each equation, $b$ is the number of black tiles (and $p$ could be used instead as the patio number) and $w$ is the number of white tiles in the patio.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w = 2b + 6$</td>
<td>(Beth and class)</td>
</tr>
<tr>
<td>$w = 2(b + 2) + 2$</td>
<td>(Faith and Damien)</td>
</tr>
<tr>
<td>$w = 3(b + 2) - b$</td>
<td>(Devon and Phoebe)</td>
</tr>
</tbody>
</table>

**Fig. 3.5.** List of rules for determining the number of white tiles in any patio

**Analyzing the Case of Darcy Dunn**

Although we could identify many aspects of the instruction in Ms. Dunn's classroom that may have contributed to her students' opportunities to learn mathematics, we will focus our attention specifically on her use of the five practices. In subsequent chapters, we will analyze a broader set of actions that, in combination with the five practices, help account for the success of the lesson. We begin by considering the five practices and whether there is evidence that the teacher engaged in some or all of these practices. Then we consider how Ms. Dunn's use of the practices may have enhanced her students' opportunities to learn.
Evidence of the five practices

As we indicated in chapter 2, determining clear and specific mathematical goals for the lesson and selecting a task that aligns with the goals are the foundation on which the five practices are built. Hence, Darcy Dunn's identification of the three mathematical ideas that she wanted her students to learn (lines 5–10) and her selection of a task that had the potential to reach these goals (fig. 3.1 and lines 11–14) positioned her to use the five practices model effectively.

Anticipating

Because the vignette focuses primarily on what happened during a classroom episode, we have limited insight into the planning in which Darcy Dunn engaged prior to the lesson and the extent to which she anticipated specific solutions to the task. However, the fact that she wanted students to know that the rate of change of a linear function can be highlighted in different representational forms suggests that she had considered the possibilities for solving the task by using a table, an equation, and a graph (lines 7–10). In addition, Darcy's decision to begin the next class with a discussion of a table and a graph suggests that she considered these approaches and their usefulness in accomplishing her goal for the lesson. We might also argue that Darcy's goal to have students recognize that there are different but equivalent ways of writing an explicit rule that defines the relationship between two variables (lines 5–7) suggests that she had probably considered different rules for relating white and black tiles before she ever set foot in the classroom.

Monitoring

Ms. Dunn monitored students working individually and in their small groups (lines 27–37). Through this monitoring she was able to determine the approaches that specific students were using (lines 28–34), ask questions to help students make progress on the task (lines 34–35), and identify what students were struggling with (lines 31–34). Her monitoring of the students' work provided the information that she needed about their mathematical thinking to modify her lesson to meet their needs and to make a decision about which strategies and solutions to focus on during the discussion. Specifically, several students were having trouble connecting verbal descriptions with symbolic rules, and as a result Ms. Dunn decided to work on this translation issue with the entire class (lines 35–37).

Selecting

By referring to notes that she had made during the monitoring process (lines 27–37), Ms. Dunn knew which students had produced specific solutions. Armed with this information, she decided to have particular students (Beth, Faith, and Devon) present approaches to the task that would lead to different symbolic rules, thus providing students with additional experience in moving between verbal and symbolic notation. In addition, she decided that she wanted students to consider the table that Tamika had built (lines 126–30) so that they could see that the rate of change was +2 (i.e., that the number of white tiles increased by 2 as the patio number increased by 1). This would highlight the constant rate of change, one of her goals for the lesson (line 5).
**Sequencing**

Ms. Dunn selected Beth as the first presenter, since her strategy had been used by several groups and therefore was likely to be one to which other students in the class could readily relate (lines 38-40). In addition, she wanted to give Beth a chance to participate actively and publicly in class (lines 40-42) since it had been several days since she had done so. By selecting Beth, Ms. Dunn was able to both highlight a popular strategy and make sure that she was providing her students with equitable opportunities to demonstrate competence.

Although it might appear that Ms. Dunn’s selection of Faith as the second presenter (lines 77-78) was a case of picking a volunteer, the fact that Ms. Dunn asked for a second method and then consulted her notes before selecting Faith (lines 75-78) suggests that the selection was strategic: Ms. Dunn was looking to see who among the volunteers had produced the strategy that she wanted to have presented. The strategy presented by Faith was a reasonable second choice since it was similar to the first strategy in that it counted only the white tiles and used the idea that there were two groups of tiles to be counted (Beth counted two groups—on the right and left side; Faith counted two groups—on the top and the bottom).

Ms. Dunn selected Devon to be the third presenter (line 97) because he had used an approach that was different from the others that had been presented up to this point, in that it focused on finding the area of the entire rectangular region and then subtracting the area of the black tiles to find the area of the white tiles. Hence, the strategy was different from the other two strategies in initially counting all the tiles, counting three groups rather than two groups, and using subtraction rather than addition. We might conclude that this strategy was not widely used within the class, and presenting other strategies first validated the thinking of the majority of students in the class and left them open to considering an alternative approach. In addition, presenting this strategy gave students access to an approach that could be useful on future tasks (lines 101-102).

Darcy’s decision to present three solutions that were all verbal descriptions of the relationship between black and white tiles depicted in the diagram seems reasonable in light of the fact that students struggled to move from verbal descriptions to explicit rules. Working on the translation from word to written descriptions to symbols as a class provided students with additional support for representing quantities abstractly, a skill that would be critical as they moved forward in their study of functions.

In addition to the verbal or visual strategies described by Beth, Faith, and Devon, Darcy also planned to have Tanika discuss the table that she had created. She intended to use the table to highlight the rate of change (the ratio of the increase in patio number to the increase in white tiles) and to connect this with the diagram and equation.

**Connecting**

Through the questions that Ms. Dunn asked during the discussion and the ways in which she pressed students to clarify what they had done and why, she helped students make connections with the mathematical ideas that were the target of her instruction. Specifically, Ms. Dunn indicated that she wanted her students to be able recognize that (1) linear functions grow at a constant rate, (2) there are different but equivalent ways of writing an explicit rule that defines the relationship between two variables, and (3) the rate of change of a linear function can be highlighted in different representational forms.
Although the students struggled to write explicit rules on their own (goal 2), Ms. Dunn pressed them to translate the verbal descriptions given by Beth, Faith, and Devon into symbolic rules after each presentation (lines 67–69; 89–90; 118–19). By using the students' verbal descriptions, supported by diagrams, as a starting point, the teacher was able to help students achieve one of her goals for the lesson.

In addition, throughout the lesson, Ms. Dunn used the work produced by students to highlight the connections among different representations (goal 3). During each presentation, Ms. Dunn encouraged the students to connect the verbal description to the diagram through the use of drawings of patios that she provided and later to symbolic rules, as previously described.

Although the teacher did not make explicit connections among student solutions in the lesson, her discussion of Timika's table the following day would position the class to connect the +2 in the table (the successive difference in the number of white tiles with each new patio) with Beth's verbal description and the related equation, as well as with the growth pattern for this function (goal 1). In addition, James's question to Phoebe (lines 120–21) provided an opportunity for students to connect the approaches used by Devon and Faith. Finally, the homework assignment given at the end of class would challenge students to consider how three different approaches could all be correct, although they looked quite different. This task could spark the notion that all three equations are equal to \( t_0 = 2f + 6 \), produce the same output for the same input, and can be linked visually to the diagram of the patio.

**Relating the five practices to learning opportunities**

Did Darcy Dunn's use of the practices contribute to her students' learning? Although we have no direct evidence of what individuals in the class learned, we see a group of students who appear to be engaged in the learning process. Over the course of the lesson, the teacher involved eight different students in substantivist ways. Ms. Dunn repeatedly targeted key ideas related to the goals of the lessons—writing explicit rules and connecting representations—as she guided her class in discussing three different solutions in depth. The final question that she gave for homework (lines 141–45) provided individual students with an opportunity to make sense of what had transpired during class and to make connections that would provide the teacher with insight into their thinking. Although the idea of a constant rate of change (goal 1) and that rate of change manifested itself differently across representations (goal 3) were not explicitly highlighted, the work that was done prepared the teacher and her students to explore these ideas in subsequent lessons.

The five practices gave the teacher a systematic approach to thinking through what her students might do with the task and how she could use their thinking to accomplish the goals that she had set. Although we analyzed the practices in action—what the teacher did during the lesson—we argue that to do what she did during the lesson, the teacher must have thought it all through before the lesson began. We will explore how to engage in such planning in subsequent chapters.

**Conclusion**

Darcy Dunn avoided a show-and-tell session in which solutions are presented in succession without much rhyme or reason, often obscuring the point of the lesson. By carefully considering the story
line of her lesson—what she wanted to accomplish mathematically and how different strategies and
representations would help her get there—she was able to question her students skillfully and position
them to make key points. So, with the lesson always firmly under her control, the teacher was able to
build on the work produced by students, carefully guiding them in a mathematically sound direction.

Consider, by contrast, the Leaves and Caterpillar vignette discussed in the introduction and
chapter 1. Although the students in Mr. Crane’s class used a range of interesting approaches, what
the students were supposed to learn from the sequence of presentations was not clear, other than
that “the problem could be solved in many different ways.” The students took no clear mathematical
message with them from this experience.

As we noted in chapter 1, the five practices build on each other, working in concert to support
the orchestration of a productive discussion. It is the information gained from engaging in one
practice that positions the teacher to engage in the subsequent practice. For example, you can’t
select solutions to be presented if you aren’t aware of what students have produced (you need to
monitor to be able to select and sequence); you can’t make connections across strategies and to
the mathematical goal of the lesson if you haven’t first selected and sequenced strategies in a way
that will help you make your point. In the next two chapters, we explore the five practices in more
depth, building on the descriptions provided in chapter 1.